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A Noisy Nutrient Induced Instability in Phytoplankton Blooms

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A B S T R A C T

The study showed the dynamics of phytoplankton blooms by a simple nutrient-phytoplankton model. A mathematical modeling with small but rapid fluctuations of nutrient shows a major modification in the blooming scenario of the system. We have analysed the condition of bifurcation of the system by expressing the autonomous kinetic equations as Li´enard oscillator which shows a very sensitivity to the noise and onset of chaos due to the bifurcation condition of the system

Introduction

Phytoplankton plants are mostly microscopic in size and unicellular (Edwards and Brindly, 1999; Pal and Choudhury, 2014). It is observed that phytoplankton grows rapidly in water, e.g., lakes, rivers, oceans etc. This growth of phytoplankton which is termed as phytoplankton bloom may be seasonal or sporadic (Griffiths, 1939; Berman et al., 1995). It is commonly believed that zooplankton grazes this phytoplankton to control their population. There are many models based on nutrient-phytoplankton-zooplankton (NPZ) model (Steele and Henderson, 1981; McCauley and Murdoch, 1987). Here we shall not discuss about the phytoplankton bloom which occurs regularly every year, but we shall consider only those phytoplankton blooms which are sporadic both in time and space. It is seen that there may be rapid and massive bloom of phytoplankton. For example, some algae may rise to several orders of magnitude and followed by sudden fall. We also know that many phytoplankton species are toxic. Blooming of such phytoplankton is known as Harmful Algae Blooom (HAB) (Hallegraeff, 1993; Anderson et al., 2008). So a rapid rise in their population may affect other species e.g., zooplankton and fish as their toxins mobilize up through the food chain. It may cause human poisoning via consumption of contaminated food. Control of the blooming of phytoplankton is possible in various ways. We can consider zooplankton as a grazer to control the population of phytoplankton. It is called top-down model (Truscott and Brindley, 1994a,b).

Also we can say that virus induced mortality (Beltrami and Carroll, 1994) can control phytoplankton bloom. The third option is that the control of nutrient can check the phytoplankton population. Such type of model is known as bottom-up model (Obrien, 1974; Hupperta et al., 2005) where concentration of nutrients control the initiation and demise of the bloom.

In this paper we shall consider the bottom-up model for population control of phytoplankton. It is seen that availability of nutrient in water causes blooming of phytoplankton. Our study is based on the fluctuation of the nutrient concentration. We introduce a noise term in the nutrient rate equation and study the influence of fluctuation of the concentration of nutrient on the evolution of phytoplankton. A very small change in the nutrient concentration leads to no bloom to bloom condition. The stochastic process is well described by a auxiliary Hamiltonian in the weak noise limit. Also we show that the autonomous kinetic equations of nutrient and phytoplankton can be expressed as Li´enard type oscillator (Ghosh and Ray, 2014). We have analysed the condition of bifurcation of this model. In this work we have shown the nature of stability of system under the influence of noise and its effect on the consequent blooming. This work was organised as follows: in section 2 we discuss bottom-up model with the introduction of noise term. In section 3 we express the kinetic equations in the Li´enard form to discuss about the bifurcation conditions of the system. Finally in section 4 we conclude the paper.

The Bottom Up Model

Fluctuation of Nutrient

A general model which describes nutrient - phytoplankton growth is given by Huppert et al., (2004)

[redacted]

where the variable N describes the concentration of the limiting nutrient in units of (mg) solute (m^{-3}) water and P the

[redacted]

phytoplankton biomass in units of (kg) solute (m^{-3}). It is assumed that external nutrient inputs flow into the system at a constant rate I in units of (mg day^{-1} m^{-3}), and that the time t is measured in days. The per capita phytoplankton uptake rate of nutrients is governed by the function g(N), for which several choices are possible. The model assumes that nutrients are lost from the water column at the rate e(N). The function f(P) represents the per capita mortality rate of the phytoplankton and also assumes the rate at which these cells sink out of the water column.

We choose the function g(N) = N. In this case the gross uptake rate g(N)P is given by the usual Lotka-Voltera bilinear term (NP). We choose e(N) = 0, i.e., there is no loss of nutrient. Control of nutrient affects the growth of phytoplankton. We, therefore, consider small but rapid fluctuation of nutrient (Ghosh and Ray, 2014) and rewrite Eqs.(1, 2) as where the dynamical system was driven by the weak white noise _(t) whose mean and

variance can be expressed as such that $\sqrt{D} \ll$

1. The Fokker-Planck Equation for probability distribution function W(N, P, t) corresponding to the Langevin description Eqs.(3, 4) can be written as

[redacted]

$$\frac{dW}{dt} = -\frac{\partial}{\partial N}(f_N W) - \frac{\partial}{\partial P}(f_P W) + D\frac{\partial^2 W}{\partial N^2} \quad (6)$$

In weak noise limit LtD→0, W(N, P, t) can be described by a WKB approximation of the Fokker Planck equation (6) of the form W(N, P, t) = W0(N, P, t) exp[S(N, P, t)/D]. Here W0 is a prefactor and S(N, P, t) is the classical action satisfying Hamilton Jacobi equation which can be solved by auxiliary Hamilton’s equation of motion

$$\frac{dN}{dt} = I - \alpha NP - 2p_N \quad (7)$$

$$\frac{dP}{dt} = \beta NP - P\{(P-a)^3 - bP + c\} \quad (8)$$

$$\frac{dp_N}{dt} = \alpha P(p_N - p_P) \quad (9)$$

$$\frac{dp_P}{dt} = \alpha N p_N - \{\alpha N - (P-a)^3 - 3P(P-a)^2 + 2bP - c\}p_P \quad (10)$$

where we have written the mortality rate function (Huppert et al., 2004)

$$f(P) = (P-a)^3 - bP^2 + c \quad (11)$$

The auxiliary Hamiltonian is given by

[redacted]

[redacted]

where, the origin of auxiliary dynamical variable pN and pP is the fluctuation of the nutrient and phytoplankton respectively. We do a numerical analysis of the Eqns. (7 – 10). We take following values to do the calculations: _ = 0.1, _ = 5, I = 0.005, a = 2, b = 1.85, c = 8. The initial conditions for the case of no bloom are N0 = 1.2 and P0 = 0.05 and for the bloom N0 = 1.3 and P0 = 0.05, here N0 and P0 are the values of N and P at time t = 0. Also we use the the noise term pN (0) = 10−25 and and pP (0) = 10−25, where (0) stands for the initial time t = 0. In Figure 1 we draw the nullclines dN/dt = 0 (dashed curve) and dP/dt = 0 (solid curve). The intersections of P-nullcline and N-nullcline give the equilibrium point. In our analysis we introduce a noise term (t) and we have seen that the system is very sensitive to this noise. The phaseplot (Fig. 2) and phytoplankton plot (Fig. 3) are similar to that obtained by Huppert et al (Huppert et al., 2004). But here we do an analysis with the introduction of a noise term and observe that the system is very sensitive to the noise.

With the choice of the noise term which is ∼

10−25, we found that the system do not get enough time to reach the equilibrium. In order that the system reaches the equilibrium the noise term should be as low as 10−90. If the value of the noise is larger than this, the system gets uncontrolled before it reaches the equilibrium state which is represented by the dark circle in Figure 2. We can say that fluctuation of the nutrient concentration has measureable effect on the evolution of the system. If the fluctuation is large the system does not reach the equilibrium state.

We also draw curves (Figures 4, 5) by varying different parameters keeping the condition of bloom i.e., N(0) = 1.3 and P(0) = 0.05. Also we take pN(0) = pP (0) =

10−25. These curves are drawn to show that though the concentration of nutrient satisfies the blooming condition, there may not be any bloom. The other parameters must have the correct value so that there is a bloom. Table 1 gives a list of values of the parameters which shows that bloom is possible for the right choice of the different parameters. The dashed curve is drawn with values N(0) = 1.2, which is the condition of no bloom. It is included here to compare with other curves of no bloom. Also the blue dot-dashed curve is drawn with value of N(0) = 1.3, which is the condition of bloom.

Sensitive dependence of noise in Phytoplankton Blooms

We draw the time evolution of the noise terms pN and pP (Figure 6). Here we have taken the initial value of pN(0) = pP (0) = 10−25. They show small value for some

time but after t ∼ 8 both the noises grow

abruptly. Also the evolution of Phytoplankton shows abrupt rise after t = 11 (Figure 7).

These curves show that noise affects the system. In the next chapter we do a bifurcation analysis of the autonomous kinetic equations to show the dependence of noise.

Bifurcation Analysis

The autonomous kinetic Eqns. (1, 2) are two coupled nonlinear equations relating the dependence of phytoplankton on nutrient. We are trying to study the evolution phytoplankton by varying the concentration of nutrient and other parameters. In section (2) we study the evolution of nutrient by introducing a small but rapid fluctuation of nutrient and found that the system is very sensitive to the noise. In order to explain this type behavior we try to find out whether the system allow any bifurcation or not. Here we shall show that the kinetic Eqns. (1, 2) can be described as Li´enard oscillator (Ghosh and Ray, 2014; Messias and Messias, 1995).

Here we define the following linear transformations as

u = (1 + I) N, z = N + P (14)

such that

z˙ = u (15)

This transformation allows the reverse set as

$$N = \frac{u}{1+I}, \quad P = z - \frac{u}{1+I} \tag{16}$$

follows:

Following Eqs (14) – (16) we obtain the

$$\begin{aligned}\ddot{z} &= (c-a^3)I - I(9a+4z)z^2 - \frac{\alpha}{(1+I)^4}(9a-16z)\dot{z}^4 - \frac{\dot{z}^5}{(1+I)^5} + I(6a^2-2b+\alpha-\beta)z \\ &- \frac{\dot{z}^2}{(1+I)^2}(9aI+a^3\alpha+c\alpha) - \frac{\alpha z^2\dot{z}^2}{(1+I)^2}(27a-62) - \frac{\dot{z}^3}{(1+I)^3}(4I+6a^2\alpha-2b\alpha-27a\alpha z \\ &+ 24\alpha z^2) + \frac{z\dot{z}}{(1+I)^2}(12I+12a^2\alpha-4b\alpha+\alpha^2-\alpha\beta) + \frac{\dot{z}}{1+I}(18aI+a^3\alpha-c\alpha) \\ &- \frac{z^2\dot{z}}{1+I}(12I+6a^2\alpha-2b\alpha+9az\alpha+\alpha^2-\alpha\beta)\end{aligned} \tag{17}$$

equation for ¨z as

The steady states zs are given by the solution of the following equation

$$4z_s^3 - 9az_s^2 + (6a-2b+\alpha-\beta)z_s - a^3 = 0 \tag{18}$$

Introducing the perturbation variable _ as (z − zs) we finally obtain the following equation

$$\ddot{\xi} + F(\xi, \dot{\xi})\dot{\xi} + G(\xi) = 0, \quad (19)$$

where

$$\begin{aligned}
F(\xi, \dot{\xi}) &= [a^3 z_s \alpha - 6a^2(I + z_s^2\alpha) + 2b(I + z_s^2\alpha) + 9a(2Iz_s + z_s^2\alpha) - z_s\{12Iz_s + \alpha(c + 4z_s^2 \\
&+ z_s - z_s\beta)\}]/(1 + I) + [a^3\alpha - 24Iz_s - 12a^2 z_s\alpha + 9a(2I + 3z_s^2\alpha) - \alpha\{c + 2z_s(8z_s^2 \\
&- 2b + \alpha - \beta)\}]\xi/(1 + I) + [12I + \alpha(6a^2 - 2b - 27az_s + 24z_s^2 + \alpha - \beta)]\xi^2/(1 + I) \\
&+ (9a - 16z_s)\alpha\xi^3/(1 + I) - 4\alpha\xi^4/(1 + I) \\
&+ [12Iz_s - a^3\alpha + 12a^2 z_s\alpha - 9a(i + 3z_s^2\alpha) + \alpha\{c - z_s(4b - 16z_s^2 - \alpha + \beta)\}]\dot{\xi}/(1 + I)^2 \\
&+ [12I + \alpha(12a^2 - 4b - 54az_s + 48z_s^2 + \alpha - \beta)]\xi\dot{\xi}/(1 + I)^2 \\
&- [4I + (6a^2 - 2b - 27az_s + 24z_s^2)\alpha]\dot{\xi}^2/(1 + I)^3 - (9a - 16z_s)\alpha\xi\dot{\xi}^2/(1 + I)^3 \\
&- 24\alpha\xi^2\dot{\xi}^2/(1 + I)^3 - (9a - 16z_s)\alpha\dot{\xi}^3/(1 + I)^4 + 16\alpha\xi\dot{\xi}^3/(1 + I)^4 - 4\alpha\dot{\xi}^4/(1 + I)^5,
\end{aligned} \quad (20)$$

$$G(\xi) = I(6a^2 - 2b - 18az_s + 12z_s^2 + \alpha - \beta)\xi - I(9a - 12z_s)\xi^2 + 4I\xi^3 \quad (21)$$

Eq. (19) assumes the form of the Linard oscillator. $F(0, 0) < 0$ gives the bifurcation condition for the oscillator.

$$\begin{aligned}
F(0, 0) &= [a^3 z_s\alpha - 6a^2(I + z_s^2\alpha) + 2b(I + z_s^2\alpha) + 9a(2Iz_s + z_s^2\alpha) - z_s(12Iz_s \\
&+ \alpha(c + 4z_s^2 + z_s\alpha - z_s\beta)\}]/(1 + I)
\end{aligned} \quad (22)$$

Substituting the values a = 2, b = 1.85, c = 8, _ = 0.1, _ = 5, I = 0.005 we get the real root of zs to be 3.35114 and the corresponding value of F(0, 0) = −0.171246. That is F(0, 0) < 0. So the system satisfies bifurcation condition (Ghosh and Ray, 2014). Bifurcations are defined as abrupt change in the phase portrait due to a change of the parameter (Hubbard and West, 1995). First we find that a small change in nutrient concentration determine whether there will be a bloom or no bloom of phytoplankton (Huppert et al., 2004). Also other parameter change (Table 1) shows such type of behavior.

Introduction of fluctuations of nutrients also exhibits instability of the system. We found that the system does not get enough time to reach the equilibrium state if the noise is greater than a certain value. It is extremely sensitive to noise. Such abrupt change in the system is due the fact that the system shows bifurcation which ultimately leads to chaos.

In conclusion, we studied the bloom of phytoplankton using bottom-up model. In this study the introduction of noise in the kinetic equation shows that the system is very sensitive to the noise. For large value of noise the system does not reach the equilibrium. Actually the noise itself grows rapidly with time which causes a chaos in the system. We rewrite the kinetic equations as a Li´enard equation and we found that the system satisfy the bifurcation condition. As we found that a small change in the initial concentration shows the unusual feature of bloom and no bloom of phytoplankton. The change of other parameter also shows such behavior. Introduction of noise term also shows such unusual behavior. The observation of this unusual behavior of the system can be interpreted by the bifurcation analysis. As the system satisfies bifurcation condition, we can say that the sensitive dependence of noise is due to the presence of bifurcation point in the system which causes instability in the system that ultimately leads to chaos.

Table.1 Bloom and no bloom condition

α	β	I	a	b	c	Remarks
0.1	5	0.005	2	1.85	8	Blue dotdashed curve, bloom condition
0.15	5	0.005	2	1.85	8	Red dotted curve, no bloom
0.1	5	0.005	2	1.6	8	Black solid curve, no bloom

Figure.1 Plot of nullclines: Dashed curve for dN/dt = 0 and solid curve for dP/dt = 0

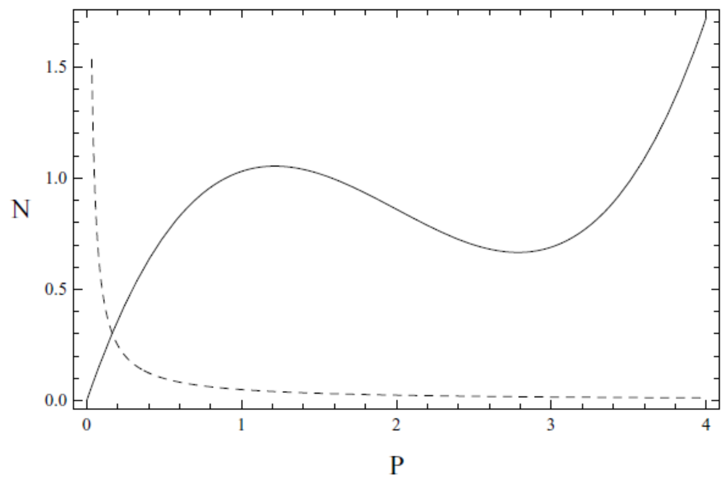


Figure.2 Phase plot of nutrient versus phytoplankton: Solid curve shows the bloom of phytoplankton and dashed curve shows no bloom situation. Dark disk is the equilibrium position which is the interaction of P−nullcline and N−nullcline (Figure 1)

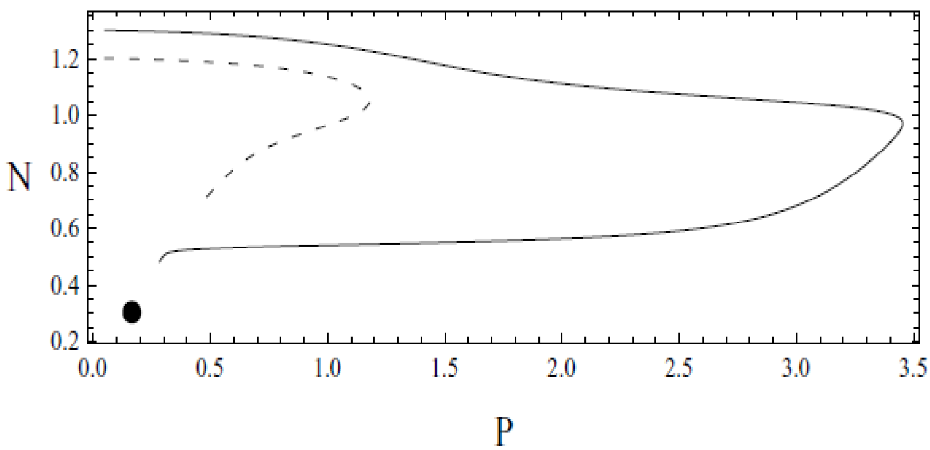


Figure.3 Growth of phytoplankton with time: Solid curve shows phytoplankton bloom and the dashed curve shows the case of no bloom

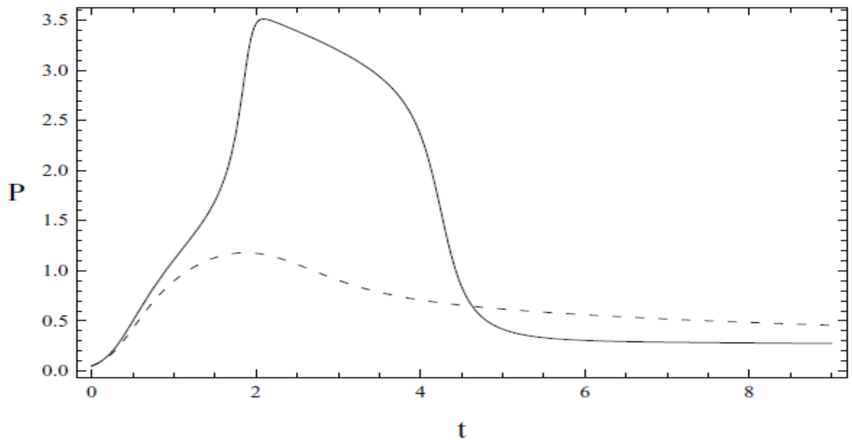


Figure.4 Phaseplot of nutrient versus phytoplankton showing bloom and no bloom for different values of the parameters: Description of the curves are listed in Table 1 and also in the text

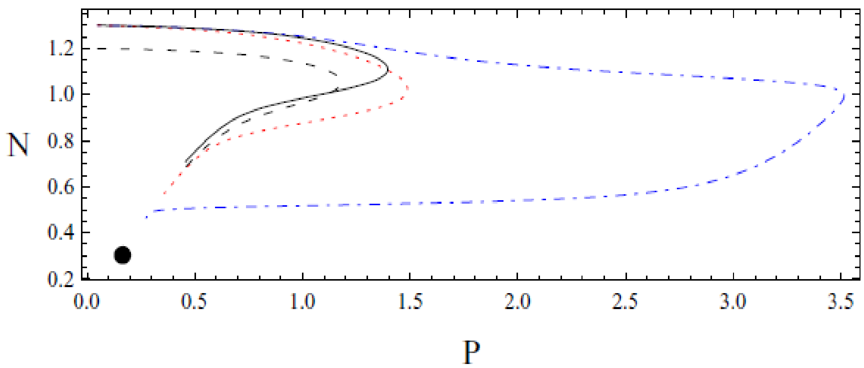


Figure.5 Plot of phytoplankton with time showing bloom and no bloom for different values of the parameters: Description of the curves are listed in Table 1 and also in the text

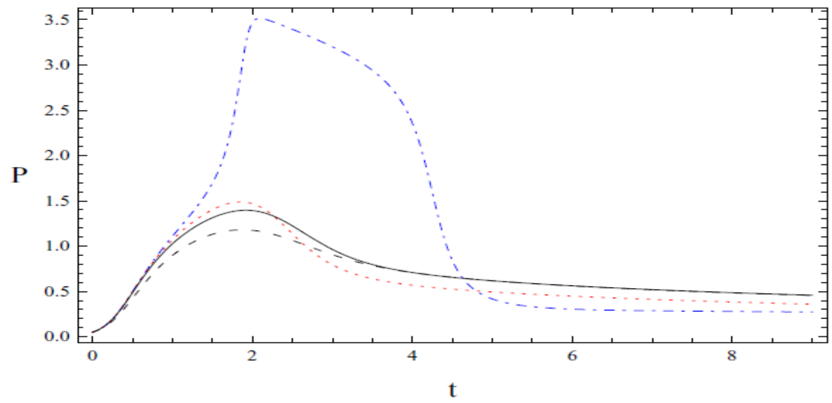


Figure.6 Growth of noise with time: Solid curve shows growth of pN and the dashed curve for Pp

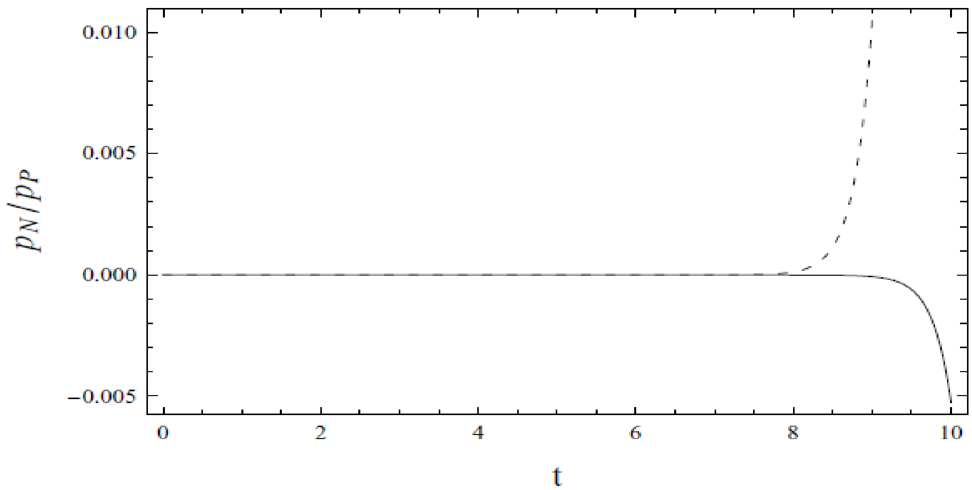
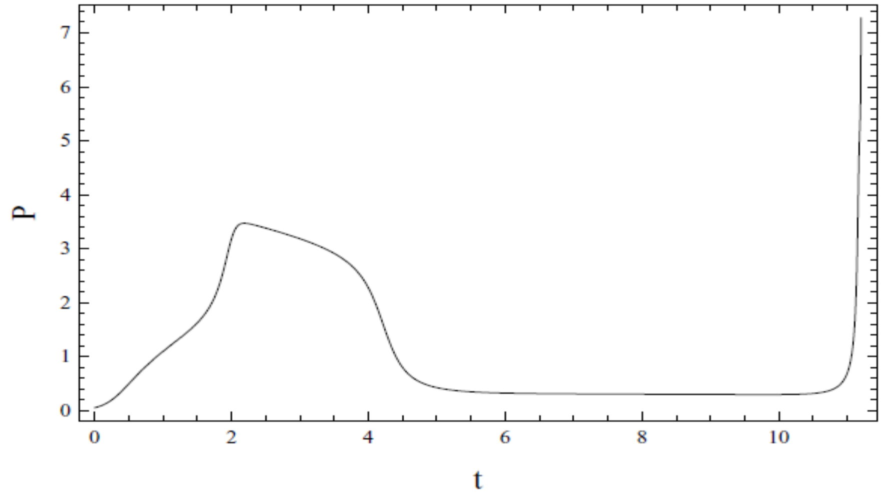


Figure.7 phytoplankton grows indefinitely after some time leading to chaos



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